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AN EXAMPLE IN AFFINE PI-RINGS

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ABSTRACT

An example is given of a prime p.i. ring R finitely generated over a field, with a prime ideal Q which is not finitely generated as a two-sided ideal of R.

In 1971 the first author raised the question [1, p. 349, p. 381] whether prime ideals in rings of generic matrices must be finitely generated (as two-sided ideals). Since every affine prime p.i. ring is a homomorphic image of a generic matrix ring, the following example provides a negative answer to this question.

Let F be any field, and let $A = F[x, y, z]$, the polynomial ring over F in three commuting indeterminates. Let $I = yA + zA$, and let P be any nonzero principal prime ideal of A lying in I^2 (e.g., $P = (y^2 - z^3)A$). Our example is a subring of the ring $M_2(A)$ of 2×2 matrices over A. Let $\{E_{ij} \mid i, j = 1, 2\}$ be the matrix units of $M_2(A)$.

Let

$$R = \begin{pmatrix} F + I^2 & I \\ I & F[x] + I^2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} P & P \\ P & P \end{pmatrix}.$$

Note that R contains the ideal $M_2(I^2)$ of $M_2(A)$. Thus, R is prime. Observe that $F[x] + I^2$ is a finitely generated F-algebra, say $F[x] + I^2 = F[b_1, \cdots, b_k]$. Also, I is finitely generated as a module over $F[x] + I^2$, say with generators $c_1, \cdots, c_n$. Then R is generated as an algebra over F by $\{E_{22}, b_1E_{22}, \cdots, b_kE_{22}, c_1E_{12}, \cdots, c_nE_{12}, c_1E_{21}, \cdots, c_nE_{21}\}$.

Since Q is a prime ideal of $M_2(A)$ not containing $M_2(I^2)$, it is easy to see that $Q = Q \cap R$ is a prime ideal of R. It remains to show that Q is not finitely generated as a two-sided ideal of R.

Let

$$J = \begin{pmatrix} IP & P \\ P & P \end{pmatrix},$$

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which is the two-sided ideal of R generated by the subset $\left(\begin{smallmatrix} 0 & P \\ P & P \end{smallmatrix}\right)$ of Q. Observe that the 1,2, the 2,1, and the 2,2 components of R act trivially on Q/J on either side, and

$$Q/J \cong \begin{pmatrix} P/IP & 0 \\ 0 & 0 \end{pmatrix}.$$

Since P is principal, $P/IP \cong A/I \cong F[x]$, with infinite dimension as a vector space over F. Hence P/IP cannot be finitely generated as a module over $F + I^2$ (the action of I^2 being trivial). But $F + I^2$ is the only part of R acting nontrivially on Q/J. Consequently, Q/J requires infinitely many generators as an R, R-bimodule, and the same must be true of Q.

REMARKS. (1) The 2,2 component of R is affine, but its 1,1 component is not even Noetherian. R/Q is not Noetherian, as its 1,1 component is not Noetherian.

(2) The classical Krull dimension of R is 3. This is known to be the minimum possible Krull dimension for such an example.

(3) In considering the other prime ideals Q' of R, one can show that Q' is not finitely generated if and only if $Q' = R \cap M_2(P')$, where P' is a prime ideal of A not containing I^2, such that $(I^2 \cap P')/I(I \cap P')$ is an infinite dimensional F-vector space.

REFERENCE

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